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Weighted scale-free networks with stochastic weight assignments

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We propose a model of weighted scale-free networks incorporating a stochastic scheme for weight assignments to the links, taking into account both the popularity and fitness of a node. As the network grows, the weights of links are driven either by the connectivity with probability p or by the fitness with probability $1-p$. Numerical results show that the total weight exhibits a power-law distribution with an exponent σ that depends on the probability p. The exponent σ decreases continuously as p increases. For $p=0$, the scaling behavior is the same as that of the connectivity distribution. An analytical expression for the total weight is derived so as to explain the features observed in the numerical results. Numerical results are also presented for a generalized model with a fitness-dependent link formation mechanism.

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Many complex systems, including social, biological, physical, economic, and computer systems, can be studied using network models in which the nodes represent the constituents and links or edges represent the interactions between constituents [1,2]. In random graphs [3,4] as well as in the small-world networks [5–7], the connectivity distribution $P(k)$, which is defined as the probability that a randomly selected node has exactly k edges, shows exponential decay. However, empirical studies on many real networks showed that $P(k)$ exhibits a power-law behavior in the tail [1,2]. Networks with power-law connectivity distributions are called *scale-free* (SF) networks. Examples of SF networks include the World Wide Web [8–10], scientific citations [11], cells [12,13], the web of actors [14], and the web of human sexual contacts [15]. The first model of SF networks was proposed by Barabási and Albert (BA) [16]. In BA networks, two important ingredients are included, namely, the networks are continuously *growing* by adding in new nodes as time evolves, and the newly added nodes are *preferentially attached* to the highly connected nodes. The idea of incorporating preferential attachment in a growing network has led to proposals of a considerable number of models of SF networks [17–23].

In most growing network models, all the links are considered equivalent. However, many real systems display different interaction strengths between nodes. In systems such as the social acquaintance network [24], the web of scientists with collaborations [25] and ecosystems [26], links between nodes may be different in their influence. Therefore, real systems are best described by weighted growing networks with links of nonuniform strengths. Only recently, a class of models of weighted growing networks was proposed by Yook, Jeong, Barabási, and Tu (YJBT) [27]. In the basic YJBT model of weighted scale-free (WSF) networks, both the topology and the weight are driven by the connectivity according to the preferential attachment rule as the network grows. It was found that the total weight distribution follows a power law $P(w)\sim w^{-\sigma}$, with an exponent σ different from the connectivity exponent γ. The difference in the exponents is a result of strong logarithmic corrections, and asymptotically (i.e., in the long time limit) the weighted and unweighted models are identical [27].

In real systems, one would expect that a link's weight and/or the growth rate in the number of links of a node depend not only on the "popularity" of the node represented by the connectivity, but also on some intrinsic quality of the node. The intrinsic quality can be collectively represented by a parameter referred to as the "fitness" [28,29]. Besides popularity, the competitiveness of a node may depend, taking for example a node being an individual in a certain community, on personality, survival skills, character, etc. A newly added node may take into account one of these factors in their decision on making connections with existing nodes and on the importance of each of the established links. Clearly, there is always a spectrum of personality among the nodes and therefore a distribution in the fitness. While one may argue that factors determining the popularity may overlap with those in fitness, it is not uncommon that popularity is not the major factor on the importance of a connection. For example, we often hear that a popular person may actually have very few good friends, and an influential and powerful figure in a network may often be someone very difficult to work with. In this Rapid Communication, we generalize the WSF model of YJBT to study the effects of fitness. In our model, the weights assigned to the newly added links are determined stochastically either by the connectivity with probability p or by the fitness of nodes with probability $1-p$. The scaling behavior of the total weight distribution is found to depend sensitively on the weight assignment mechanism through the parameter p.

The topological structure of our model follows that of the BA model of SF networks [16]. A small number (m_0) of nodes are created initially. At each time step, a new node j with m ($m\leqslant m_0$) links is added to the network. These m links will connect to m preexisting nodes according to the preferential attachment rule that the probability Π_i of an existing node i being selected for connection is proportional to the total number of links k_i that node i carries, i.e.,

$$\Pi_i = \frac{k_i}{\sum_l k_l}. \quad (1)$$

The procedure creates a network with $N = t + m_0$ nodes and mt links after t time steps. Geometrically, the network displays a connectivity distribution with a power-law decay in the tail with an exponent $\gamma = 3$, regardless of the value of m [16,30].

A weighted growing network is constructed by assigning weights to the links as the network grows. To incorporate a fitness-dependent weight assignment mechanism, a fitness parameter η_i is assigned to each node [28,29]. The fitness η_i is chosen randomly from a distribution $\rho(\eta)$, which is assumed to be a uniform distribution in the interval $[0,1]$ for simplicity. With probability p, each newly established link $j \leftrightarrow i$ is assigned a weight w_{ji} $(= w_{ij})$ given by

$$w_{ji} = \frac{k_i}{\sum_{\{i'\}} k_{i'}}, \quad (2)$$

where $\Sigma_{\{i'\}}$ is a sum over the m nodes to which the new node j is connected. With probability $1-p$, w_{ji} is determined by the fitness through

$$w_{ji} = \frac{\eta_i}{\sum_{\{i'\}} \eta_{i'}}. \quad (3)$$

In Eqs. (2) and (3), w_{ji} is normalized so that $\Sigma_{\{i'\}} w_{ji'} = 1$ [27]. For $p = 1$, our model reduces to the YJBT model with entirely connectivity-driven weights [27]. For $p = 0$, the weights are driven entirely by the fitness. For $0 < p < 1$, the present model provides a possible stochastic weight assignment scheme in which a newly added node, e.g., representing some newcomer into a web, considers either the popularity or the fitness of its connected neighbor in assigning w_{ji}.

We performed extensive numerical simulations on the model, with networks up to $N = 5 \times 10^5$ nodes with $m = m_0 = 5$. For each value of p, results are obtained by averaging over ten independent runs. First, we study the total weight distribution $P(w)$, which is defined as the probability that a randomly selected node has a total weight w. The total weight of a node i is given by the sum of the weights of all links connected to it, i.e., $w_i = \Sigma_j w_{ij}$. Figure 1 shows that $P(w)$ behaves as $P(w) \sim w^{-\sigma}$, with an exponent σ that decreases from the value of 3 at $p = 0$ continuously as p increases. For $p = 1$, $\sigma = 2.4$, a result in agreement with that of YJBT [27]. For $p = 0$, $\sigma = 3$ $(= \gamma)$ showing that $P(w)$ follows the same scaling behavior as $P(k)$. YJBT found that the scaling behavior of $P(w)$ depends strongly on m [27] in their model. Here, we found that the m dependence persists for all $p > 0$. Only when $p = 0$, σ becomes independent of m.

It is also interesting to study the dynamical behavior of the total weight $w_i(\eta_i, t)$ of some node i with fitness η_i. Figure 2 shows that $w_i(\eta_i, t)$ grows in time as a power law with a p-dependent exponent δ. For $p > 0$, $\delta > \beta$, where $\beta = 1/2$ is the exponent characterizing the dynamical behavior of the connectivity $k_i(t)$ [16]. For $p = 0$, $w_i(\eta_i, t)$ shows the same scaling behavior as $k_i(t)$ with $\delta = \beta = 1/2$. For $0 < p < 1$, δ also depends on the node's fitness η_i. Thus, the total weight actually shows a multiscaling dynamical behavior in the range $0 < p < 1$ [28].

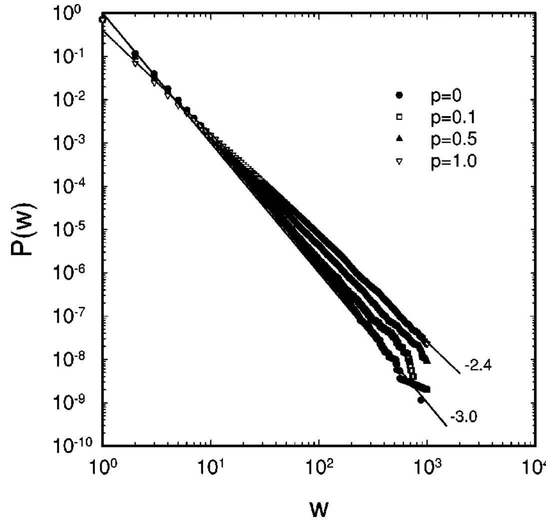

FIG. 1. The weight distribution $P(w)$ as a function of the total weight w on a log-log scale for different values of $p = 0, 0.1, 0.5, 1.0$. The two solid lines are guides to the eye corresponding to the exponents $\sigma = 2.4$ and 3.0, respectively.

The probability distribution $P(w_{ij})$ of the weights w_{ij} is also worth investigating. To suppress statistical fluctuations, Fig. 3 shows the cumulative distribution, $P(x > w_{ij})$, instead of $P(w_{ij})$, on a log-linear scale. For $p = 0$, $P(x > w_{ij})$ decays exponentially in the tail. Recall that $P(w)$ and $w_i(\eta_i, t)$ show identical behavior as $P(k)$ and $k_i(t)$ for $p = 0$, respec-

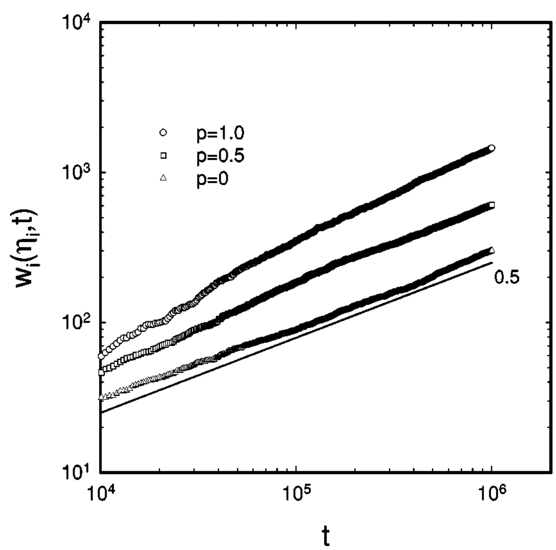

FIG. 2. The total weight $w_i(\eta_i, t)$ of a randomly selected node i with fitness η_i $(= 0.75)$ as a function of time t on a log-log scale for different values of $p = 0, 0.5, 1.0$. The solid line is a guide to the eye corresponding to an exponent $\sigma = 0.5$.

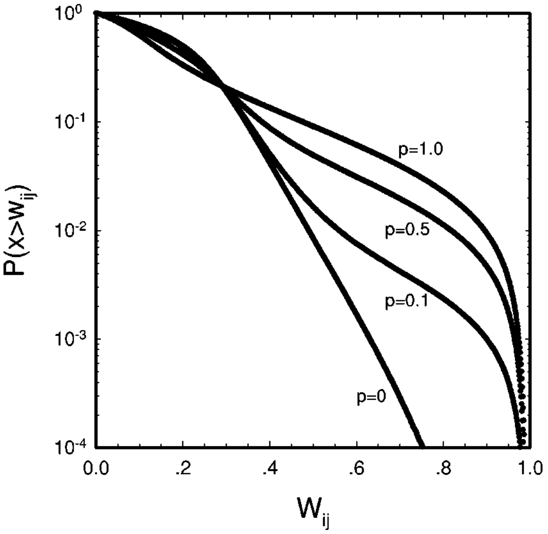

FIG. 3. The cumulative distribution $P(x>w_{ij})$ of the weights of individual links as a function of w_{ij} on a log-linear plot for different values of $p=0,0.1,0.5,1.0$.

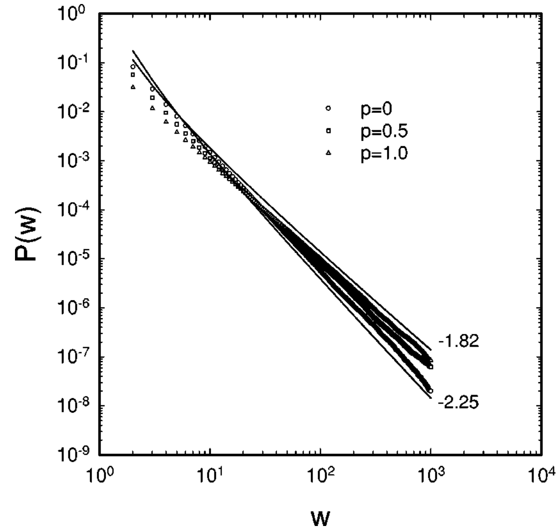

FIG. 4. The weight distribution $P(w)$ as a function of the total weight w on a log-log scale for different values of $p=0,0.5,1.0$ in a model with fitness-dependent link formation mechanism. The two solid lines are plotted according to the form of $P(w)\sim w^{-\sigma'}/\ln w$, with exponent σ' taking the values 1.82 and 2.25, respectively.

tively, and the latter two quantities are not sensitive to the weight assignment scheme. Here, $P(x>w_{ij})$ shows an exponentially decaying behavior, implying that the weighted and unweighted models are not entirely identical even for $p=0$. For $p>0$, the tail deviates from an exponentially decaying form and decays faster as p increases. For $p=1$, we recover the results in the YJBT model [27].

To understand the different behavior between $w_i(\eta_i,t)$ and $k_i(t)$ [as well as between $P(w)$ and $P(k)$] found in numerical simulations, we derive an analytical expression for the total weight $w_i(\eta_i,t)$ of a node i with fitness η_i at time t. Following YJBT [27], $w_i(\eta_i,t)$ can be expressed as

$$w_i(\eta_i,t)=1+\int_{t_i^0}^{t}\int_{m}^{\infty}\int_{0}^{1}\tilde{P}_i(m,t')w_{ji}(\eta_l,k_l)\varrho(k_l)\times\rho(\eta_l)d\eta_l dk_l dt', \qquad (4)$$

where $\tilde{P}_i(m,t)$ is the probability that node i is selected for connection to a new node j at time t for given m and it is related to Π_i in Eq. (1) by a factor of m. Here, t_i^0 is the time at which the node i has been added to the system. $w_{ji}(\eta_l,k_l)$ is the weight assigned to the link. $\varrho(k)$ and $\rho(\eta)$ are the probability distributions of k and η, respectively. According to Eqs. (2) and (3), the weight $w_{ji}(\eta_l,k_l)$, on the average, can be written as

$$w_{ji}(\eta_l,k_l)=p\frac{k_i}{k_i+k_l}+(1-p)\frac{\eta_i}{\eta_i+\eta_l} \qquad (5)$$

for the simple case of $m=2$. Generalization to arbitrary value of m is straightforward.

From the connectedness of the SF model, $\tilde{P}_i(m,t)$, $\varrho(k)$, and $k_i(t)$ are given by [30,27] $\tilde{P}_i(m,t)=m\Pi_i=k_i(t)/2t$, $\varrho(k)=mk^{-2}$, and $k_i(t)=m\sqrt{t/t_i^0}$, respectively. Substituting these relationships into Eq. (4) and noticing that $\rho(\eta)$ is assumed to be a uniform distribution in the interval [0,1], Eq. (4) becomes

$$w_i(\eta_i,t)\simeq\left[p+2(1-p)\eta_i\ln\frac{1+\eta_i}{\eta_i}\right]k_i(t)-\frac{1}{4}p\left[\left(\ln\frac{4t}{t_i^0}\right)^2-4\ln 2\ln\frac{t}{t_i^0}\right]+C, \qquad (6)$$

where C is an integration constant. Equation (6) implies that the different scaling behavior in $w_i(\eta_i,t)$ and $k_i(t)$ are results of the logarithmic correction term, which can be tuned by the parameter p. For $p\to 0$, Eq. (6) gives $w_i(\eta_i,t)\sim 2\eta_i\ln[(1+\eta_i)/\eta_i]k_i(t)$, leading to the same scaling behavior of $w_i(\eta_i,t)$ and $k_i(t)$, as observed in the simulation results. For $p=1$, the dynamical behavior of $w_i(\eta_i,t)$ deviates most from that of $k_i(t)$ [27]. For arbitrary m, $w_i(\eta_i,t)$ follows a similar form with m dependence coming into the second term on the right-hand side of Eq. (6).

Our model can be generalized to allow for a fitness-dependent link formation mechanism [28,29]. In the basic model with fitness [28], the probability Π_i depends on both the connectivity and fitness through

$$\Pi_i=\frac{\eta_i k_i}{\sum_l \eta_l k_l}. \qquad (7)$$

To study the effects of fitness, we study a generalization of our model by replacing Eq. (1) by Eq. (7) for link formation, while keeping Eqs. (2) and (3) for weight assignments. The connectivity distribution follows a generalized power law [28] with an inverse logarithmic correction of the form

$P(k) \sim k^{-\gamma'}/\ln k$, with $\gamma' = 2.255$. Figure 4 shows the numerical results for the total weight distribution $P(w)$ for three different values of $p = 0, 0.5$, and 1. It is found that $P(w)$ follows the same generalized power-law form as $P(k)$, but with a different exponent σ' that depends on p. For $p > 0$, $\sigma' < \gamma'$. Only for $p = 0$, $P(w)$ and $P(k)$ have the same exponent of $\sigma' = 2.25 \sim \gamma'$. The cumulative distribution $P(x > w_{ij})$ of weights is similar to those shown in Fig. 3.

In summary, we proposed and studied a model of weighted scale-free networks in which the weights are stochastically determined by the connectivity of nodes with probability p and by the fitness of nodes with probability $1 - p$. The model leads to a power-law probability distribution for the total weight characterized by an exponent σ that is highly sensitive to the probability p. A similar result was also found in a generalized model with a fitness-dependent link formation mechanism. An expression relating the total weight and the total connectivity of a node was derived and the result was used to explain the features observed in numerical simulations. In conclusion, we note that although the distributions $P(w)$ and $P(k)$ carry different exponents σ and γ for $p > 0$ in the models studied here, $P(w)$ still follows a power law, i.e., it has the same functional form as $P(k)$. However, one would expect that in many complex real systems, even the functional forms of $P(w)$ and $P(k)$ may be different. It remains a challenge to construct simple and yet nontrivial models that give different behavior for the geometrical connection and the extent of connectivity in a network.

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